Perturbative analysis of generally nonlocal spatial optical solitons

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In analogy to a perturbed harmonic oscillator, we calculate the fundamental and some other higher order soliton solutions of the nonlocal nonlinear Schödinger equation (NNLSE) in the second approximation in the generally nonlocal case. Comparing with numerical simulations we show that soliton solutions in the second approximation can describe the generally nonlocal soliton states of the NNLSE more exactly than that in the zeroth approximation. We show that for the nonlocal case of an exponential-decay type nonlocal response the Gaussian-function-like soliton solutions cannot describe the nonlocal soliton states exactly even in the strongly nonlocal case. The properties of such nonlocal solitons are investigated. In the strongly nonlocal limit, the soliton's power and phase constant are both in inverse proportion to the fourth power of its beam width for the nonlocal case of a Gaussian function type nonlocal response, and are both in inverse proportion to the third power of its beam width for the nonlocal case of an exponential-decay type nonlocal response.

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I. INTRODUCTION

Since Snyder and Mitchell's pioneering work [1], spatial solitons propagating in nonlocal nonlinear media have been investigated experimentally and theoretically in a variety of configurations and material systems. It is theoretically indicated that stable spatial bright (dark) soliton states can be admitted in self-focus (self-defocus) weekly nonlocal media [2] and Gaussian-function-like bright soliton states can be admitted in self-focused strongly nonlocal media [1,3]. It has been shown theoretically that nonlocality drastically modifies the interaction of dark solitons by inducing a long-range attraction between them, thereby permitting the formation of stable dark soliton bound states [4]. The propagation properties of light beams in the presence of losses in the strongly nonlocal case are different from that in the local case [5]. By considering the special case of a logarithmic type of nonlinearity and a Gaussian-function type nonlocal response, the dynamics of beams in partially nonlocal media [6] and the propagation of incoherent optical beams [7] are analytically studied. By using the variational principle, the propagation properties of a solitary wave in nonlinear nonlocal medium with a power function type nonlocal response are studied [8]. The modulational instability of plane waves in nonlocal Kerr media [9,10] and the stabilizing effect of nonlocality [11] have been studied. The analogy between parametric interaction in quadratic media and nonlocal Kerr-type nonlinearities can provide a physically intuitive theory for quadratic solitons [12]. Some properties of the strongly nonlocal solitons (SNSs) and their interaction are greatly different from that in the local case, e.g., two coherent SNSs with π phase difference attract rather than repel each other [1], the phase shift of the SNS can be very large compared with the local soliton with the same beam width [3], and the phase shift of a probe beam can be modulated by a pump beam in the strongly nonlocal case [13]. Employing a Gaussian ansatz and using a variational approach, the evolution of a Gaussian beam in the substrongly nonlocal case is studied [14]. Recently it is experimentally shown that solitons in the nematic liquid crystal (NLC) are SNSs [15,18]. The team of Assanto has developed a general theory of spatial solitons in the NLC that exhibit nonlinearity with an arbitrary degree of an effective nonlocality and established an important link between the SNS and the parametric soliton [15–18]. They also experimentally investigated the role of the nonlocality in transverse modulational instability (MI) in the NLC [16,17] and observed the optical multisoliton generation following the onset of spatial MI [19]. The interaction of SNSs has been experimentally demonstrated [20,21], and the possibility of all-optical switching and logic gating with SNSs in the NLC has been discussed [22].

However, the theoretical studies on the spatial nonlocal soliton are mostly focused on the strongly nonlocal case [1,3,5,13,15,18] and the weekly nonlocal case [2]. There is a lack of study on the moderate nonlocal case. On the other hand, even though a convenient method has been introduced in Refs. [3,5,13,14] to study the propagation of light beams in the strongly nonlocal case or even in the substrongly nonlocal case, to employ this method efficiently the nonlocal response function must be twice differentiable at its center. As will be shown this method cannot deal with the nonlocal case of an exponential-decay type nonlocal response function that is not differentiable at its center. In this paper, in analogy to a perturbed harmonic oscillator, we calculate the fundamental and some other higher order soliton solutions of the nonlocal nonlinear Schrödinger equation (NNLSE) in the second approximation in the generally nonlocal case. Our method presented here can deal with the nonlocal case of an exponential-decay type nonlocal response function. Numerical simulations conform that the soliton solution in the second approximation can describe the generally nonlocal soliton states of NNLSE more exactly than that in the zeroth approximation. It is shown that for the nonlocal case of an exponential-decay type nonlocal response the Gaussian-function-like soliton solutions cannot describe the fundamental soliton states of the NNLSE exactly even in the strongly nonlocal case, that is greatly different from the case of a Gaussian-function type nonlocal response. The properties of

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such nonlocal solitons are investigated. The functional dependence of such nonlocal soliton's power and phase constant on its beam width is greatly different from that of the local soliton. Furthermore this functional dependence for the nonlocal case of a Gaussian-function type nonlocal response greatly differs from that of an exponential-decay type nonlocal response. In particular in the strongly nonlocal limit, the nonlocal soliton's power and phase constant are both in inverse proportion to the fourth power of its beam width for the nonlocal case of a Gaussian-function type nonlocal response, and are both in inverse proportion to the third power of its beam width for the nonlocal case of an exponential-decay type nonlocal response.

II. THE FUNDAMENTAL GENERALLY NONLOCAL SOLITON SOLUTION IN THE SECOND APPROXIMATION

Let us consider the (1+1)-D dimensionless nonlocal nonlinear Schödinger equation (NNLSE) [2,3,7–10,12]

$$i\frac{\partial u}{\partial z}+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}+u\int_{-\infty}^{+\infty}R(x-\xi)|u(\xi,z)|^2d\xi=0, \qquad (1)$$

where $u(x,z)$ is the complex amplitude envelop of the light beam, x and z are transverse and longitude coordinates respectively, $R(x)>0$ is the real symmetric nonlocal response function, and

$$n(x,z)=\int_{-\infty}^{+\infty}R(x-\xi)|u(\xi,z)|^2d\xi \qquad (2)$$

is the light-induced perturbed refractive index.

As indicated in Ref. [3], if $R(x)$ is twice differentiable at $x=0$ and the second derivative $R''(0)<0$, and if the characteristic nonlocal length is one order of the magnitude larger than the beamwidth of the soliton, the NNLSE (1) can be simplified to the following strongly nonlocal model (SNM):

$$i\frac{\partial u}{\partial z}+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}+u\int_{-\infty}^{+\infty}\left[R_0+\frac{R_0''}{2}(x-\xi)^2\right]|u(\xi,z)|^2d\xi=0, \qquad (3)$$

where $R_0=R(0)$ and $R_0''=R''(0)$. For example, for the Gaussian-function type nonlocal response function $R(x)=1/(w_0\sqrt{\pi})\exp(-x^2/w_0^2)$, when the characteristic nonlocal length w_0 is one order of the magnitude larger than the beam width, the SNM (3) can describe the NNLSE (1) very well [3]. However, as will be shown, when the characteristic nonlocal length and the beamwidth are in the same order of the magnitude, the SNM (3) cannot describe the NNLSE (1) very well. The SNM (3) cannot deal with the generally nonlocal case. Further more, for the exponential-decay type nonlocal response function $R(x)=1/(2w_0)\exp(-|x|/w_0)$ which is not differentiable at $x=0$, we cannot get the parameter R_0'' of the SNM (3). So the SNM (3) cannot deal with this nonlocal case of such an exponential-decay type nonlocal response.

The SNM (3) allows a Gaussian-function-like bright soliton solution

$$u_0(x,z)=A\left(\frac{1}{\pi\nu^2}\right)^{1/4}\exp\left[-\frac{x^2}{2\nu^2}-i\left(\frac{3}{4\nu^2}-R_0A^2\right)z\right], \qquad (4)$$

where

$$\frac{1}{\nu^4}=-R_0''A^2, \qquad (5)$$

and ν is the beam width of $u_0(x,z)$. The power P and the phase constant γ of $u_0(x,z)$ are given by

$$P=\int_{-\infty}^{+\infty}|u(x,t)|^2dx=A^2, \qquad (6)$$

$$\gamma=R_0A^2-\frac{3}{4\nu^2}, \qquad (7)$$

respectively.

In this paper, we define the degree of nonlocality by the ratio of the characteristic nonlocal length to the beamwidth of the light beam and use the phrase "*generally nonlocal case*" to refer to the nonlocal case where the degree of nonlocality is larger than one and less than ten. For the Gaussian-function type nonlocal response function and the soliton solution (4), the degree of nonlocality is w_0/ν. The larger of w_0/ν, the stronger of the nonlocality. In fact for a given type of nonlocal response, soliton solutions with the same degree of nonlocality can be described in the same way. This can be clarified by taking transformations

$$\bar{x}=\frac{x}{\kappa} \quad \bar{z}=\frac{z}{\kappa^2} \quad \bar{u}=\kappa u \quad \bar{R}=\kappa R. \qquad (8)$$

Under these transformations, the form of NNLSE (1) keeps invariant and the degree of nonlocality keeps invariant too. If we set κ equal to the characteristic nonlocal length of $R(x)$, the characteristic nonlocal length of $\bar{R}(\bar{x})$ will be scaled to unity and the degree of nonlocality will be determined only by the beam width of $\bar{u}(\bar{x},\bar{z})$. In this case the less of the beamwidth of $\bar{u}(\bar{x},\bar{z})$, the stronger of the nonlocality. On the other hand we may also set κ equal to the beamwidth of $u(x,z)$. If we do this, the degree of nonlocality will be determined only by the characteristic nonlocal length of $\bar{R}(\bar{x})$. The larger the characteristic nonlocal length of $\bar{R}(\bar{x})$, the stronger the nonlocality. In this paper, the characteristic nonlocal length of $R(x)$ and the beamwidth of $u(x,z)$ are not scaled to unity.

For the soliton state $u(x,z)$, we have $|u(-x,z)|^2=|u(x,z)|^2$ and $|u(x,z)|=|u(x,0)|$. So for the soliton state $u(x,z)$, by defining

$$V(x)=-\int_{-\infty}^{+\infty}R(x-\xi)|u(\xi,z)|^2d\xi, \qquad (9)$$

the NNLSE (1) reduces to

$$i\frac{\partial u}{\partial z}+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}-V(x)u=0. \qquad (10)$$

Taking the Taylor's expansion of $V(x)$ at $x=0$, we obtain

$$V(x) = V_0 + \frac{1}{2\mu^4}x^2 + \alpha x^4 + \beta x^6 + \cdots, \tag{11}$$

where

$$V_0 = V(0), \tag{12a}$$

$$\frac{1}{\mu^4} = V^{(2)}(0), \tag{12b}$$

$$\alpha = \tfrac{1}{4!}V^{(4)}(0), \tag{12c}$$

$$\beta = \tfrac{1}{6!}V^{(6)}(0). \tag{12d}$$

As will be shown, in the generally nonlocal case and the strongly nonlocal case the parameter μ can be viewed as the beamwidth of the soliton, and when $x<\mu$, the terms αx^4 and βx^6 are one and two orders of magnitude smaller than the term $x^2/(2\mu^4)$, respectively. That indicates the effects of αx^4 and βx^6 on the soliton are considerably small compared with the effect of $x^2/(2\mu^4)$ in the generally nonlocal case. Furthermore in the generally nonlocal case, the effects of the x^8 power term and the other higher power terms of the Taylor's series of $V(x)$ on the soliton are far smaller than the effects of the three lower power terms. For convenience sake we will neglect such higher power terms in the following discussions and simply adopt

$$V(x) = V_0 + \frac{1}{2\mu^4}x^2 + \alpha x^4 + \beta x^6. \tag{13}$$

However, as the degree of nonlocality decreases the effects of αx^4, βx^6 and other higher power terms become larger and larger, and when the characteristic nonlocal length is comparable with or less than the beamwidth of the soliton, the x^8 power term and other higher power terms are no longer negligible. For such cases we must take the higher power terms of the Taylor's series of $V(x)$ into account.

In the generally nonlocal case, substitution of Eq. (13) into Eq. (10) yields

$$i\frac{\partial u}{\partial z} = \left[-\frac{1}{2}\frac{\partial^2}{\partial x^2} + V_0 + \frac{1}{2\mu^4}x^2 + \alpha x^4 + \beta x^6\right]u. \tag{14}$$

Taking a transformation

$$u(x,z) = \psi_n(x)\exp[-i(\varepsilon_n + V_0)z], \tag{15}$$

we arrive at

$$\left[-\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2\mu^4}x^2 + \alpha x^4 + \beta x^6\right]\psi_n = \varepsilon_n\psi_n, \tag{16}$$

where the index $n=0,1,2,\ldots$ is the order of the soliton solution, in particular $n=0$ corresponding to the fundamental soliton solution and $n=1$ corresponding to the second order soliton solution and so on. Even though Eq. (16) takes the form of the stationary Schrödinger equation, the parameters μ,α,β are dependent on the soliton solution $\psi_n(x)$.

If $\alpha=0$ and $\beta=0$, Eq. (16) reduces to the well-known stationary Schrödinger equation for a harmonic oscillator. Since in the generally nonlocal case the effects of the terms αx^4 and βx^6 on the soliton are far smaller than that of the term $x^4/(2\mu^4)$, we view the terms αx^4 and βx^6 as perturbations in the process of solving Eq. (16). Following the perturbation method presented in any textbook about quantum mechanics (for example, [23]), we get for the fundamental soliton solution in the second approximation

$$\begin{aligned}\psi_0(A,\alpha,\beta,x) \approx{}& A\left(\frac{1}{\pi\mu^2}\right)^{1/4}\exp\left(-\frac{x^2}{2\mu^2}\right)\\ &\times\left[1 + \alpha\left(\frac{9\mu^6}{16} - \frac{3\mu^4}{4}x^2 - \frac{\mu^2}{4}x^4\right)\right.\\ &+ \alpha^2\left(-\frac{1247\mu^{12}}{512} + \frac{141\mu^{10}}{64}x^2 + \frac{53\mu^8}{64}x^4\right.\\ &\left.+ \frac{13\mu^6}{48}x^6 + \frac{\mu^4}{32}x^8\right) + \beta\left(\frac{55\mu^8}{32} - \frac{15\mu^6}{8}x^2\right.\\ &\left.\left. - \frac{5\mu^4}{8}x^4 - \frac{\mu^2}{6}x^6\right)\right],\end{aligned} \tag{17}$$

and

$$\varepsilon_0 \approx \frac{1}{2\mu^2} + \frac{3\mu^4\alpha}{4} - \frac{21\mu^{10}\alpha^2}{8} + \frac{15\mu^6\beta}{8}. \tag{18}$$

In Eqs. (17) and (18), if we neglect the α,α^2,β terms or neglect the α^2,β terms only, we will get for the fundamental soliton solution in the zeroth approximation or in the first approximation, respectively.

Substituting Eq. (17) into Eq. (9), we have

$$V(A,\alpha,\beta,x) = -\int_{-\infty}^{+\infty} R(x-\xi)\psi_0^2(A,\alpha,\beta,\xi)d\xi. \tag{19}$$

Keeping in mind Eqs. (12), we obtain

$$\frac{1}{\mu^4} = V^{(2)}(A,\alpha,\beta,0), \tag{20a}$$

$$\alpha = \tfrac{1}{4!}V^{(4)}(A,\alpha,\beta,0), \tag{20b}$$

$$\beta = \tfrac{1}{6!}V^{(6)}(A,\alpha,\beta,0). \tag{20c}$$

For a fixed value of parameter μ, the parameters A,α,β can be found by solving Eqs. (20). In Appendix A, we present a fixed-point method to numerically calculate these parameters based on Eqs. (20). Here and above we have formally presented the main formulas to calculate the perturbed fundamental generally nonlocal soliton solution in the second approximation.

A. The nonlocal case of a Gaussian-function type nonlocal response

As an example, let us consider the nonlocal case of a Gaussian-function type nonlocal response [3,6,7,9,11]

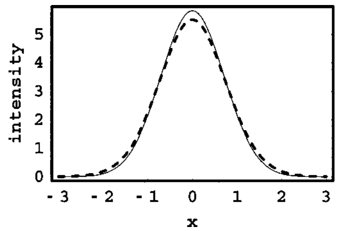

FIG. 1. The comparison between $|\psi_0(A,\alpha,\beta,x)|^2$ (dashed line) and $|\psi_0(A,0,0,x)|^2$ (solid line). Here $w_0=2$, $\mu=1$, $A=3.217$, $\alpha=-0.0487$, $\beta=0.003\,17$, and the degree of nonlocality $w_0/\mu=2$.

$$R(x) = \frac{1}{w_0\sqrt{\pi}} \exp\left(-\frac{x^2}{w_0^2}\right). \tag{21}$$

For the SNM (3) and the soliton solution (4), we can find the fundamental soliton solution for such a Gaussian-function type nonlocal response in the strongly nonlocal case

$$u_0(x,z) = \left(\frac{\sqrt{\pi}w_0^3}{2\nu^4}\right)^{1/2}\left(\frac{1}{\pi\nu^2}\right)^{1/4} e^{-(x^2/2\nu^2)-i\left(\frac{3}{4\nu^2}-\frac{w_0^2}{2\nu^4}\right)z}. \tag{22}$$

This soliton solution can describe the soliton state of the NNLSE (1) exactly in the strongly nonlocal case when the degree of nonlocality $w_0/\nu > 10$, but cannot describe the soliton state in the generally nonlocal case when $w_0/\nu \sim 2$.

In the generally nonlocal case, the fundamental soliton solution in the second approximation is described by $\psi_0(A,\alpha,\beta,x)$ in Eq. (17). As shown in Fig. 1 when $w_0=2$, $\mu=1$, and $A=3.22$, $\alpha=-0.0487$, $\beta=0.003\,17$ numerically calculated by the fixed-point method presented in Appendix A, the difference between the fundamental soliton solution in the second approximation $\psi_0(A,\alpha,\beta,x)$ and that in the zeroth approximation $\psi_0(A,0,0,x)$ is comparatively small. As a Gaussian function, the power and the beamwidth of $\psi_0(A,0,0,x)$ are given by A^2 and μ, respectively. Therefore the power and the beamwidth of $\psi_0(A,\alpha,\beta,x)$ are approximatively given by A^2 and μ, respectively, too. So in the generally nonlocal case we can approximately determine the degree of nonlocality by w_0/μ, and approximately obtain

$$V(x) \approx -\int_{-\infty}^{+\infty} \frac{1}{w_0\sqrt{\pi}} \exp\left[-\frac{(x-\xi)^2}{w_0^2}\right]\psi_0^2(A,0,0,\xi)d\xi$$
$$= -\frac{A^2}{\sqrt{\pi(\mu^2+w_0^2)}}\exp\left(-\frac{x^2}{\mu^2+w_0^2}\right), \tag{23}$$

and

$$A^2 \approx \frac{\sqrt{\pi}(1+w_0^2/\mu^2)^{3/2}}{2\mu}, \tag{24a}$$

$$V_0 \approx -\frac{(1+w_0^2/\mu^2)}{2\mu^2}, \tag{24b}$$

$$\alpha \approx -\frac{1}{4\mu^6(1+w_0^2/\mu^2)}, \tag{24c}$$

$$\beta \approx \frac{1}{12\mu^8(1+w_0^2/\mu^2)^2}. \tag{24d}$$

Using Eqs. (24) for $w_0=2$ and $\mu=1$, we can find $A \approx 3.14$, $\alpha \approx -\frac{1}{20}$, and $\beta \approx \frac{1}{300}$ that are very close to the numerically calculated values $A=3.22$, $\alpha=-0.0487$, and $\beta=0.003\,17$, and we can find $\left|\frac{\alpha x^4}{x^2/(2\mu^4)}\right| \approx \left|\frac{x^2}{10\mu^2}\right| < 0.1$ and $\left|\frac{\beta x^6}{x^2/(2\mu^4)}\right| \approx \left|\frac{x^4}{150\mu^4}\right| < 0.007$ for $x<\mu$ that are consistent with the perturbation postulate.

In the strongly nonlocal limit the degree of nonlocality $w_0/\mu \gg 1$, we have

$$A^2 \approx \frac{\sqrt{\pi}w_0^3}{2\mu^4}, \tag{25a}$$

$$V_0 \approx -\frac{w_0^2}{2\mu^4}, \tag{25b}$$

$$\alpha \approx -\frac{1}{4\mu^4 w_0^2}, \tag{25c}$$

$$\beta \approx \frac{1}{12\mu^4 w_0^4}. \tag{25d}$$

As the degree of nonlocality w_0/μ approaches infinity, the parameters α,β both approach zero, and $\psi_0(A,\alpha,\beta,x)$ approaches $\psi_0(A,0,0,x)$. In such a case a Gaussian-function-like strongly nonlocal soliton solution is obtained.

Using the NNLSE (1) as the evolution equation and using the numerical simulation method we investigate the propagation of light beams in nonlocal media with a Gaussian-function type nonlocal response. The numerical simulation method is the split-step Fourier method (SSFM) [24], the step size $\Delta z=0.01$, transversal sampling range $-10 \leqslant x \leqslant 10$, and the sampling interval $\Delta x=0.1$. With different input amplitude envelops (the initial data of numerical simulations) that are described by $u_0(x,0)$ in Eq. (22), $\psi_0(A,0,0,x)$ and $\psi_0(A,\alpha,\beta,x)$ respectively, we show the propagations of these light beams in Fig. 2. It is indicated that in the generally nonlocal case when the degree of nonlocality $w_0/\mu=2$, $\psi_0(A,\alpha,\beta,x)$ can describe the soliton state of the NNLSE (1) more exactly than $\psi_0(A,0,0,x)$ and $u_0(x,0)$ in Eq. (22). The soliton solution in the second approximation $\psi_0(A,\alpha,\beta,x)$ also can describe the soliton state of the NNLSE (1) exactly when $w_0/\mu=1$, that is shown in Fig. 3. However, when $w_0/\mu=0.5$, as indicated in Fig. 4, $\psi_0(A,\alpha,\beta,x)$ cannot describe the soliton state of the NNLSE (1) exactly. In such a case, we must take the higher power terms of the Taylor's series of $V(x)$ into account and calculate the higher order approximation. To show how exactly $\psi_0(A,\alpha,\beta,x)$ describe the fundamental soliton state, we define

$$\theta(z) = \sqrt{\frac{\int_{-\infty}^{+\infty}|u(x,z)e^{-i\phi(z)}-u(x,0)|^2dx}{\int_{-\infty}^{+\infty}|u(x,0)|^2dx}}, \tag{26a}$$

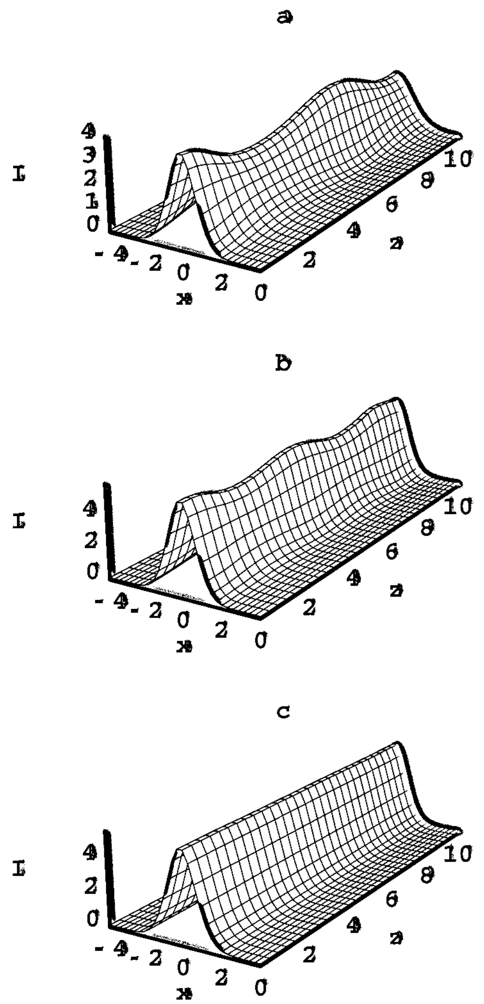

FIG. 2. The propagations of light beams in nonlocal media with a Gaussian-function type nonlocal response with different input intensity profiles that are described by (a) $|u_0(x,0)|^2$ in Eq. (22), (b) $|\psi_0(A,0,0,x)|^2$ and (c) $|\psi_0(A,\alpha,\beta,x)|^2$, respectively. Here $w_0=2$, $\mu=1$, $\nu=1$, $A=3.217$, $\alpha=-0.0487$, $\beta=0.003\,17$ and the degree of nonlocality $w_0/\mu=2$.

$$\bar{\theta}=\frac{\int_0^l \theta(z)dz}{l}, \tag{26b}$$

where $e^{i\phi(z)}$ is the phase factor of $u(x,z)$ and for the fundamental soliton $e^{i\phi(z)}=\frac{u(0,z)}{|u(0,z)|}$. For a fixed value of l, the less of $\bar{\theta}$, the more exactly $\psi_0(A,\alpha,\beta,x)$ describe the fundamental soliton state. As shown in Table I, $\psi_0(A,\alpha,\beta,x)$ can describe the fundamental soliton states exactly when $w_0/\mu>1$.

Now let us consider the properties of $\psi_0(A,\alpha,\beta,x)$. As has been shown, the beamwidth of $\psi_0(A,\alpha,\beta,x)$ is approximatively given by μ, and its power and phase constant are approximatively given by

$$P\approx A^2\approx\frac{\sqrt{\pi}(1+w_0^2/\mu^2)^{3/2}}{2\mu}, \tag{27}$$

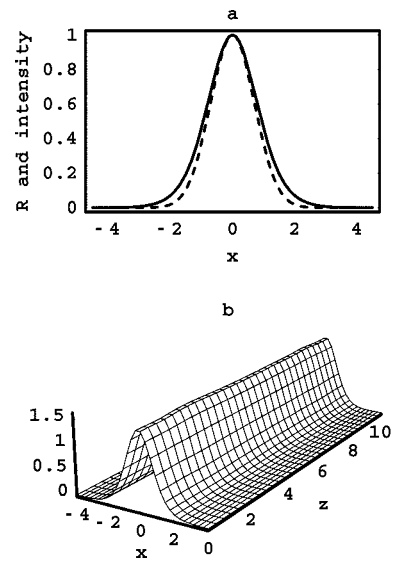

FIG. 3. (a) The comparison between $R(x)$ and $|\psi_0(A,\alpha,\beta,x)|^2$. Dashed line: $R(x)/R(0)$; solid line: $|\psi_0(A,\alpha,\beta,x)|^2/|\psi_0(A,\alpha,\beta,0)|^2$; (b) The propagation of the light beam with an input intensity profile described by $|\psi_0(A,\alpha,\beta,x)|^2$. Here $w_0=1$, $\mu=1$, $A=1.777$, $\alpha=-0.113$, $\beta=0.0177$, and the degree of nonlocality $w_0/\mu=1$.

$$\gamma=-V_0-\varepsilon_0\approx\frac{1}{2\mu^2}\left[\frac{w_0^2}{\mu^2}+\frac{3}{8(1+w_0^2/\mu^2)}+\frac{1}{64(1+w_0^2/\mu^2)^2}\right], \tag{28}$$

respectively. In the strongly nonlocal limit the degree of nonlocality $w_0/\mu\gg1$, we have

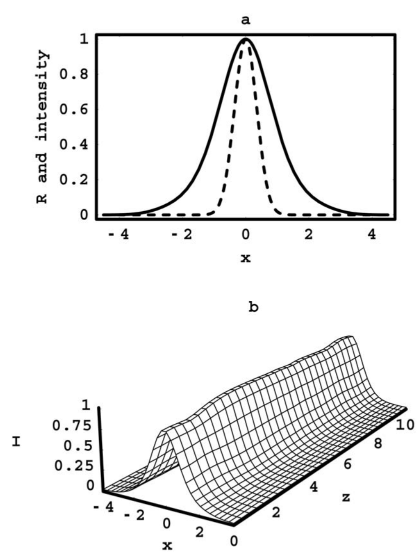

FIG. 4. (a) The comparison between $R(x)$ and $|\psi_0(A,\alpha,\beta,x)|^2$. Dashed line: $R(x)/R(0)$; solid line: $|\psi_0(A,\alpha,\beta,x)|^2/|\psi_0(A,\alpha,\beta,0)|^2$; (b) The propagation of the light beam with an input intensity profile described by $|\psi_0(A,\alpha,\beta,x)|^2$. Here $w_0=0.5$, $\mu=1$, $A=1.403$, $\alpha=-0.178$, $\beta=0.0476$, and the degree of nonlocality $w_0/\mu=0.5$.

TABLE I. Using the numerical simulation method we calculate $\bar{\theta}$ in Eqs. (26) for the nonlocal case of the Gaussian function type nonlocal response and for the nonlocal case of the exponential-decay type nonlocal response.

	$\bar{\theta}$						
ψ_0 [a]	0.020 [c]	0.0033 [d]	0.0027 [e]	0.0027 [f]	0.0026 [g]	0.0026 [h]	0.0026 [i]
ψ_1 [a]	0.044 [c]	0.013 [d]	0.013 [e]	0.013 [f]	0.013 [g]	0.013 [h]	0.013 [i]
ψ_0 [b]	0.017 [c]	0.0095 [d]	0.0078 [e]	0.0072 [f]	0.0071 [g]	0.0069 [h]	0.0068 [i]
ψ_1 [b]	0.072 [c]	0.044 [d]	0.037 [e]	0.034 [f]	0.032 [g]	0.031 [h]	0.031 [i]
ψ_2 [b]	0.36 [c]	0.20 [d]	0.17 [e]	0.15 [f]	0.14 [g]	0.14 [h]	0.13 [i]

[a] The nonlocal case of the Gaussian function type nonlocal response.
[b] The nonlocal case of the exponential-decay type nonlocal.
[c] The degree of nonlocality equal to 1.
[d] The degree of nonlocality equal to 2.
[e] The degree of nonlocality equal to 3.
[f] The degree of nonlocality equal to 4.
[g] The degree of nonlocality equal to 5.
[h] The degree of nonlocality equal to 6.
[i] The degree of nonlocality equal to 7.

$$P \approx \frac{\sqrt{\pi} w_0^3}{2\mu^4}, \tag{29}$$

$$\gamma \approx \frac{w_0^2}{2\mu^4}. \tag{30}$$

That means for a given value of characteristic nonlocal length in the strongly nonlocal case the power and the phase constant of the nonlocal soliton are both in inverse proportion to the fourth power of its beamwidth. The dependence of the power P and the phase constant γ on the beamwidth μ are shown in Fig. 5 for a given value of characteristic nonlocal length. It is indicated that Eq. (27) and Eq. (30) can describe these dependences exactly in the generally nonlocal case.

To make a comparison with the local soliton, let us consider the following local nonlinear Schrödinger equation (NLSE) [25]:

$$i\frac{\partial u}{\partial z} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + |u|^2 u = 0. \tag{31}$$

When the characteristic nonlocal length w_0 approaches zero, the Gaussian-function type nonlocal response function $R(x)$ approaches the $\delta(x)$ function, and the NNLSE (1) approaches the NLSE (31). The fundamental soliton of the NLSE (31) is given by [25]

$$u(x,z) = \frac{1}{\eta}\,\mathrm{sech}\left(\frac{x}{\eta}\right)\exp\left(i\frac{z}{2\eta^2}\right), \tag{32}$$

where η can be viewed as the beamwidth of the local soliton. The power and the phase constant of such a local soliton are given by

$$P = \int_{-\infty}^{+\infty} |u(x,t)|^2 dx = \frac{2}{\eta}, \tag{33}$$

$$\gamma = \frac{1}{2\eta^2}, \tag{34}$$

respectively. We can find the power and the phase constant of the local soliton are in inverse proportion to the first and the second power of its beamwidth, respectively. The functional dependence of the power and the phase constant of the nonlocal soliton on its beamwidth greatly differs from that of the local soliton.

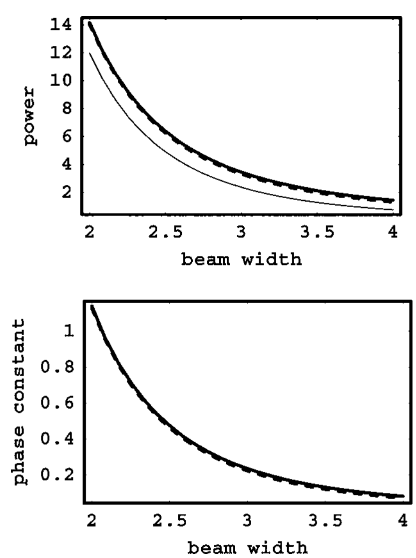

FIG. 5. The dependence of the power P and the phase constant γ on the beamwidth μ. Dashed lines are described by Eq. (27) and Eq. (30), respectively; thin solid line is described by Eq. (29); thick solid lines are directly calculated with parameters A, α, β numerically found by the fixed-point method presented in Appendix A. Here w_0=6.

B. The nonlocal case of an exponential-decay type nonlocal response

As another example, we investigate the nonlocal case where the light-induced perturbed refractive index $n(x,z)$ is governed by [4,9,15]

$$n(x,z) - w_0^2 \frac{\partial^2 n(x,z)}{\partial x^2} - |n(x,z)|^2 = 0. \qquad (35)$$

It is found that several nonlocal media, for example the nematic liquid crystal [15,18], their light-induced perturbed refractive index can be described by Eq. (35). If the size of the nonlocal media is much larger than the beamwidth of the soliton and the characteristic nonlocal length, the effect of the boundary condition on the soliton can be negligible and we can simply assume the size of the nonlocal media is infinity large. For such a case, Eq. (35) leads to

$$n(x,z) = \frac{1}{2w_0}\int_{-\infty}^{+\infty} \exp\left(-\frac{|x-\xi|}{w_0}\right)|u(\xi,z)|^2 d\xi, \qquad (36)$$

and we get the exponential-decay type nonlocal response [4,7,9–11]

$$R(x) = \frac{1}{2w_0}\exp\left(-\frac{|x|}{w_0}\right). \qquad (37)$$

Since the exponential-decay type nonlocal response function $R(x)$ is not differentiable at $x=0$, the SNM (3) cannot deal with this nonlocal case. So we have to use $\psi_0(A,\alpha,\beta,x)$ to describe the soliton state of the NNLSE (1).

For this exponential-decay type nonlocal response and the fundamental soliton state, $V(x)$ can be approximately given by

$$\begin{aligned} V(x) &\approx -\int_{-\infty}^{+\infty} \frac{1}{2w_0}\exp\left(-\frac{|x-\xi|}{w_0}\right)|\psi_0(A,0,0,\xi)|^2 d\xi \\ &= \frac{A^2}{4w_0} e^{\mu^2/4w_0^2}\left\{ e^{-x/w_0}\left[\operatorname{erf}\left(\frac{\mu}{2w_0} - \frac{x}{\mu}\right) - 1\right] \right. \\ &\quad \left. + e^{x/w_0}\left[\operatorname{erf}\left(\frac{\mu}{2w_0} + \frac{x}{\mu}\right) - 1\right]\right\}, \end{aligned} \qquad (38)$$

where

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-\xi^2} d\xi. \qquad (39)$$

Combining Eqs. (12), we get

$$A^2 \approx \frac{1/\mu}{\dfrac{\mu^2}{\sqrt{\pi}w_0^2} + \exp\left(\dfrac{\mu^2}{4w_0^2}\right)\dfrac{\mu^3}{2w_0^3}\left[\operatorname{erf}\left(\dfrac{\mu}{2w_0}\right) - 1\right]}, \qquad (40a)$$

$$V_0 \approx -\frac{A^2 \exp\left(\dfrac{\mu^2}{4w_0^2}\right)\left[1 - \operatorname{erf}\left(\dfrac{\mu}{2w_0}\right)\right]}{2w_0}, \qquad (40b)$$

$$\alpha \approx \frac{A^2\left\{\exp\left(\dfrac{\mu^2}{4w_0^2}\right)\left[\operatorname{erf}\left(\dfrac{\mu}{2w_0}\right) - 1\right] + \dfrac{2w_0}{\sqrt{\pi}\mu} - \dfrac{4w_0^3}{\sqrt{\pi}\mu^3}\right\}}{48w_0^5}, \qquad (40c)$$

$$\beta \approx \frac{A^2\left\{\exp\left(\dfrac{\mu^2}{4w_0^2}\right)\left[\operatorname{erf}\left(\dfrac{\mu}{2w_0}\right) - 1\right] + \dfrac{2w_0}{\sqrt{\pi}\mu} - \dfrac{4w_0^3}{\sqrt{\pi}\mu^3} + \dfrac{24w_0^5}{\sqrt{\pi}\mu^5}\right\}}{1440w_0^7}. \qquad (40d)$$

In the strongly nonlocal limit the degree of nonlocality $w_0/\mu \gg 1$, we obtain

$$A^2 \approx \frac{\sqrt{\pi}w_0^2}{\mu^3}, \qquad (41a)$$

$$V_0 \approx -\frac{\sqrt{\pi}w_0}{2\mu^3}, \qquad (41b)$$

$$\alpha \approx -\frac{1}{12\mu^6}, \qquad (41c)$$

$$\beta \approx \frac{1}{60\mu^8}. \qquad (41d)$$

It is worth noting that in the strongly nonlocal case the parameters α and β are free from the characteristic nonlocal length w_0. Even when the characteristic nonlocal length w_0 approaches infinity, the parameters α,β still rest on finite values and do not approach zero, and therefore $\psi_0(A,\alpha,\beta,x)$ does not approach $\psi_0(A,0,0,x)$. That greatly differs from the nonlocal case of a Gaussian-function type nonlocal response. As a result the Gaussian-function-like soliton solution $\psi_0(A,0,0,x)$ cannot describe the soliton state of the NNLSE (1) exactly even in the strongly nonlocal case, that is shown in Fig. 6. As shown in Fig. 7, $\psi_0(A,\alpha,\beta,x)$ also can describe the soliton state of the NNLSE (1) exactly when $w_0/\mu=1$. Even when $w_0/\mu=0.5$, as indicated in Fig. 8, $\psi_0(A,\alpha,\beta,x)$ can describe the soliton state of the NNLSE (1) in high quality. As indicated by the values of $\bar{\theta}$ in Table I, $\psi_0(A,\alpha,\beta,x)$

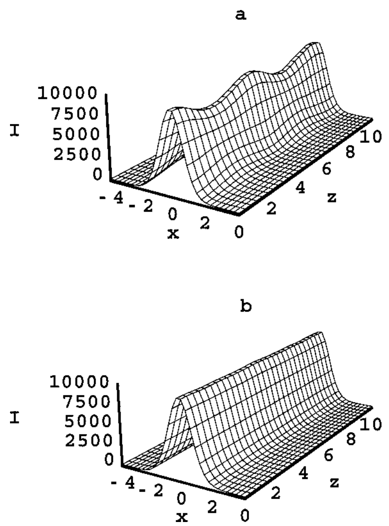

FIG. 6. The propagations of light beams with different input intensity profiles that are described by (a) $|\psi_0(A,0,0,x)|^2$ and (b) $|\psi_0(A,\alpha,\beta,x)|^2$, respectively. Here $w_0=100$, $\mu=1$, $A=138.159$, $\alpha=-0.0767$, $\beta=0.0150$ and the degree of nonlocality $w_0/\mu=100$.

can describe the fundamental soliton states of NNLSE (1) exactly in the generally nonlocal case.

In the strongly nonlocal case the soliton's power and phase constant are approximately given by

$$P \approx A^2 \approx \frac{\sqrt{\pi}w_0^2}{\mu^3}, \tag{42}$$

$$\gamma \approx -V_0 \approx \frac{\sqrt{\pi}w_0}{2\mu^3}, \tag{43}$$

respectively. For a given value of the characteristic nonlocal length, the soliton's power and phase constant are both in inverse proportion to the third power of its beamwidth in the strongly nonlocal case that differs from the nonlocal case of a Gaussian function type nonlocal response where the soliton's power and phase constant are both in inverse proportion to the fourth power of its beamwidth in the strongly nonlocal case. The dependence of the soliton's power P and phase constant γ on its beamwidth μ are shown in Fig. 9 for a given value of characteristic nonlocal length. It is indicated that Eq. (42) and Eq. (43) can describe these dependence very well in the strongly nonlocal case.

III. THE HIGHER ORDER GENERALLY NONLOCAL SOLITON SOLUTIONS IN THE SECOND APPROXIMATION

A. The nonlocal case of the Gaussian-function type nonlocal response

The second order soliton solution for the SNM (3) is given by

$$u_1(x,z) = A\left(\frac{1}{\pi\nu^2}\right)^{1/4}\frac{\sqrt{2}x}{\nu}e^{-x^2/2\nu^2-i(9/4\nu^2-R_0A^2)z}, \tag{44}$$

where

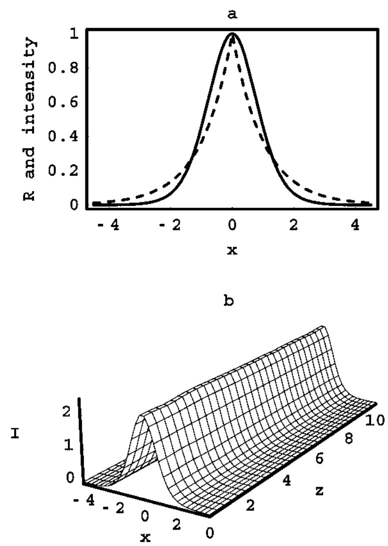

FIG. 7. (a) The comparison between $R(x)$ and $|\psi_0(A,\alpha,\beta,x)|^2$. Dashed line: $R(x)/R(0)$; solid line: $|\psi_0(A,\alpha,\beta,x)|^2/|\psi_0(A,\alpha,\beta,0)|^2$; (b) the propagation of the light beam with an input intensity profile described by $|\psi_0(A,\alpha,\beta,x)|^2$. Here $w_0=1$, $\mu=1$, $A=2.206$, $\alpha=-0.126$, $\beta=0.0280$, and the degree of nonlocality $w_0/\mu=1$.

$$\frac{1}{\nu^4} = -R_0''A^2. \tag{45}$$

The power and the beam width of $u_1(x,z)$ are given by A^2 and $\sqrt{3}\nu$, respectively. For the Gaussian-function type nonlocal response, the second order soliton solution for the SNM (3) is given by

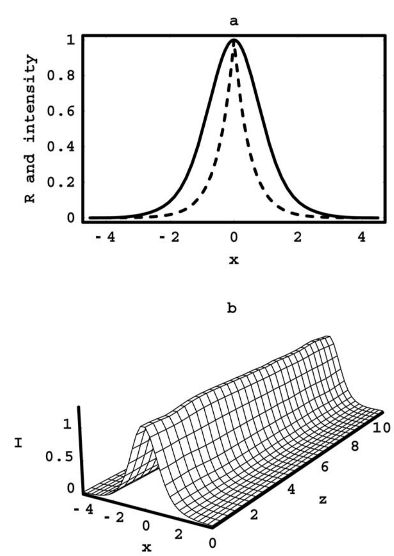

FIG. 8. (a) The comparison between $R(x)$ and $|\psi_0(A,\alpha,\beta,x)|^2$. Dashed line: $R(x)/R(0)$; solid line: $|\psi_0(A,\alpha,\beta,x)|^2/|\psi_0(A,\alpha,\beta,0)|^2$; (b) the propagation of the light beam with an input intensity profile described by $|\psi_0(A,\alpha,\beta,x)|^2$. Here $w_0=0.5$, $\mu=1$, $A=1.610$, $\alpha=-0.158$, $\beta=0.0408$, and the degree of nonlocality $w_0/\mu=0.5$.

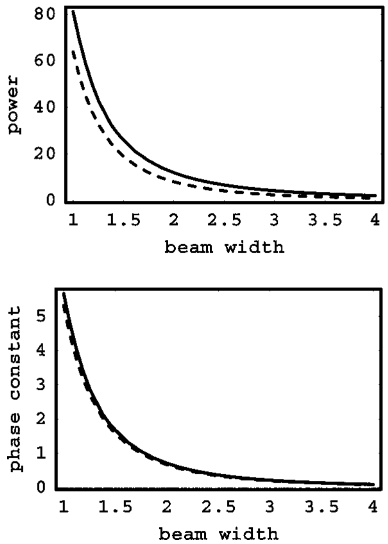

FIG. 9. The dependence of the soliton's power P and phase constant γ on its beamwidth μ. Dashed lines are described by Eq. (42) and Eq. (43), respectively; solid lines are directly calculated with parameters A, α, β numerically found by the fixed-point method presented in Appendix A. Here $w_0=6$.

$$u_1(x,z) = \left(\frac{\sqrt{\pi}w_0^3}{2\nu^4}\right)^{1/2}\left(\frac{1}{\pi\nu^2}\right)^{1/4}\frac{\sqrt{2}x}{\nu}e^{-(x^2/2\nu^2)-i([9/4\nu^2]-[w_0^2/2\nu^4])z}. \tag{46}$$

This soliton solution can describe the second order soliton state of the NNLSE (1) exactly in the strongly nonlocal case when $w_0/(\sqrt{3}\nu)>10$ but cannot describe it exactly in the generally nonlocal case when $w_0/(\sqrt{3}\nu)\sim 2$.

The second order generally nonlocal soliton solution in the second approximation is given by

$$\begin{aligned}\psi_1(A,\alpha,\beta,x) \approx{}& A\left(\frac{1}{\pi\mu^2}\right)^{1/4}\exp\left(-\frac{x^2}{2\mu^2}\right)\frac{\sqrt{2}x}{\mu}\\ &\times\left[1+\alpha\left(\frac{45\mu^6}{16}-\frac{5\mu^4}{4}x^2-\frac{\mu^2}{4}x^4\right)\right.\\ &+\alpha^2\left(-\frac{8375\mu^{12}}{512}+\frac{215\mu^{10}}{64}x^2+\frac{73\mu^8}{64}x^4\right.\\ &\left.+\frac{19\mu^6}{48}x^6+\frac{\mu^4}{32}x^8\right)+\beta\left(\frac{385\mu^8}{32}-\frac{35\mu^6}{8}x^2\right.\\ &\left.\left.-\frac{7\mu^4}{8}x^4-\frac{\mu^2}{6}x^6\right)\right],\end{aligned} \tag{47}$$

and

$$\varepsilon_1 \approx \frac{3}{2\mu^2}+\frac{15\alpha\mu^4}{4}-\frac{165\alpha^2\mu^{10}}{8}+\frac{105\beta\mu^6}{8}. \tag{48}$$

For the Gaussian-function type nonlocal response function (21), as shown in Fig. 10, the difference between $|\psi_1(A,\alpha,\beta,x)|^2$ and $|\psi_1(A,0,0,x)|^2$ is small in the generally nonlocal case. As an Hermite-Gaussian function, the power and the beamwidth of $\psi_1(A,0,0,x)$ are given by A^2 and

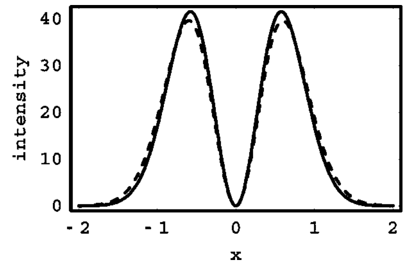

FIG. 10. The comparison between $|\psi_1(A,\alpha,\beta,x)|^2$ (dashed line) and $|\psi_1(A,0,0,x)|^2$ (solid line). Here $w_0=1.5$, $\mu=1/\sqrt{3}$, $A=7.606$, $\alpha=-0.450$, $\beta=0.005\,30$, and the degree of nonlocality $w_0/(\sqrt{3}\mu)=1.5$.

$\sqrt{3}\mu$, respectively. So the power and the beamwidth of $\psi_1(A,\alpha,\beta,x)$ are also approximatively given by A^2 and $\sqrt{3}\mu$, respectively. We can approximatively determine the degree of nonlocality by $w_0/(\sqrt{3}\mu)$ and approximatively obtain

$$\begin{aligned}V(x) &\approx -\int_{-\infty}^{+\infty}\frac{1}{w_0\sqrt{\pi}}\exp\left[-\frac{(x-\xi)^2}{w_0^2}\right]\psi_1^2(A,0,0,\xi)d\xi\\ &= -\frac{A^2}{\sqrt{\pi(\mu^2+w_0^2)}}e^{-\frac{x^2}{\mu^2+w_0^2}}\frac{2x^2/\mu^2+w_0^2/\mu^2+w_0^4/\mu^4}{(1+w_0^2/\mu^2)^2},\end{aligned} \tag{49}$$

and

$$A^2 \approx \frac{\sqrt{\pi}(1+w_0^2/\mu^2)^{5/2}}{2\mu(w_0^2/\mu^2-2)}, \tag{50a}$$

$$V_0 \approx -\frac{w_0^2(1+w_0^2/\mu^2)}{2\mu^4(w_0^2/\mu^2-2)}, \tag{50b}$$

$$\alpha \approx -\frac{(w_0^2/\mu^2-4)}{4\mu^6(1+w_0^2/\mu^2)(w_0^2/\mu^2-2)}, \tag{50c}$$

$$\beta \approx \frac{(w_0^2/\mu^2-6)}{12\mu^8(1+w_0^2/\mu^2)^2(w_0^2/\mu^2-2)}. \tag{50d}$$

In the strongly nonlocal limit the degree of nonlocality $w_0/(\sqrt{3}\mu)\gg 1$, we have

$$A^2 \approx \frac{\sqrt{\pi}w_0^3}{2\mu^4}, \tag{51a}$$

$$V_0 \approx -\frac{w_0^2}{2\mu^4}, \tag{51b}$$

$$\alpha \approx -\frac{1}{4\mu^4 w_0^2}, \tag{51c}$$

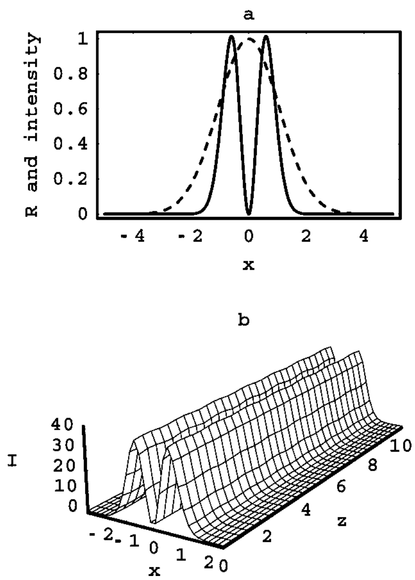

FIG. 11. (a) The comparison between $R(x)$ and $|\psi_1(A,\alpha,\beta,x)|^2$. Dashed line: $R(x)/R(0)$; solid line: $|\psi_1(A,\alpha,\beta,x)|^2/|\psi_1(A,\alpha,\beta,0.65)|^2$; (b) the propagation of the light beam with an input intensity profile described by $|\psi_1(A,\alpha,\beta,x)|^2$. Here $w_0=1.5$, $\mu=1/\sqrt{3}$, $A=7.606$, $\alpha=-0.450$, $\beta=0.005\,30$, and the degree of nonlocality $w_0/(\sqrt{3}\mu)=1.5$.

$$\beta \approx \frac{1}{12\mu^4 w_0^4}. \tag{51d}$$

As the degree of nonlocality $w_0/(\sqrt{3}\mu)$ approaches infinity, the parameters α and β approach zero, and $\psi_1(A,\alpha,\beta,x)$ approaches $\psi_1(A,0,0,x)$. Therefore in the strongly nonlocal case an Hermite-Gaussian-function-like second order soliton solution is obtained, and the power and the phase constant of $\psi_1(A,\alpha,\beta,x)$ are both in inverse proportion to the fourth power of its beamwidth. As indicated in Fig. 11 and Fig. 12, the second order soliton solution in the second perturbation $\psi_1(A,\alpha,\beta,x)$ can describe the second order soliton state of the NNLSE (1) exactly when $w_0/(\sqrt{3}\mu)=1.5$ and describe it in high quality when $w_0/(\sqrt{3}\mu)=1$. As shown by the values of $\bar{\theta}$ in Table I, $\psi_1(A,\alpha,\beta,x)$ can exactly describe the second order nonlocal soliton state in the generally nonlocal cases.

Finally since all the eigenfunctions of the harmonic oscillator can be found systematically [23], it is possible that in

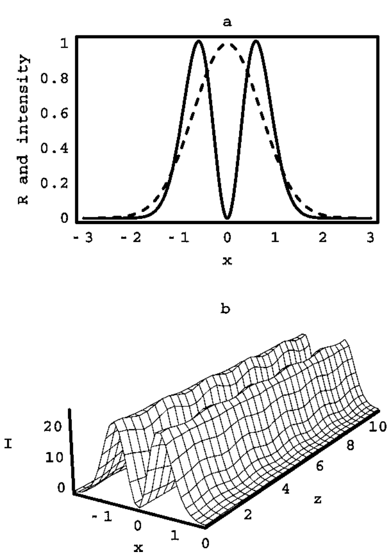

FIG. 12. (a) The comparison between $R(x)$ and $|\psi_1(A,\alpha,\beta,x)|^2$. Dashed line: $R(x)/R(0)$; solid line: $|\psi_1(A,\alpha,\beta,x)|^2/|\psi_1(A,\alpha,\beta,0.65)|^2$; (b) the propagation of the light beam with an input intensity profile described by $|\psi_1(A,\alpha,\beta,x)|^2$. Here $w_0=1$, $\mu=1/\sqrt{3}$, $A=6.133$, $\alpha=-0.492$, $\beta=0.0145$, and the degree of nonlocality $w_0/(\sqrt{3}\mu)=1$.

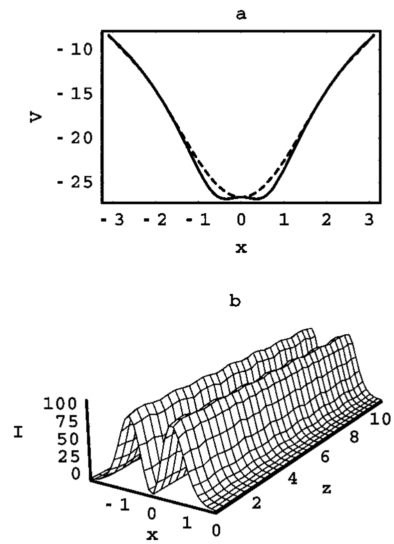

FIG. 13. (a) The comparison between $V(A,\alpha,\beta,x)$ (solid line) and $\tilde{V}(x)$ (dashed line); (b) the propagation of the light beam with an input intensity profile described by $|\psi_1(A,\alpha,\beta,x)|^2$. Here $w_0=2$, $\mu=1/\sqrt{3}$, $A=12.162$, $\alpha=-0.442$, $\beta=0.0179$, and the degree of nonlocality $w_0/(\sqrt{3}\mu)=2$.

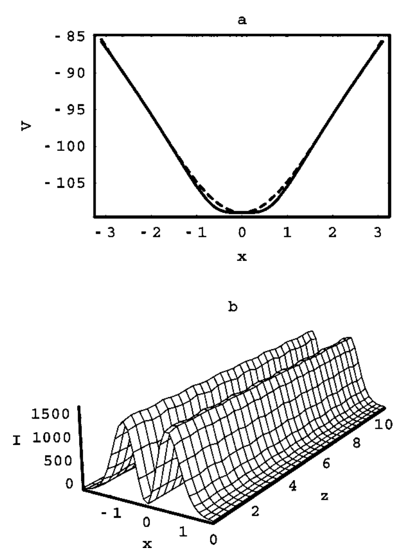

FIG. 14. (a) The comparison between $V(A,\alpha,\beta,x)$ (solid line) and $\tilde{V}(x)$ (dashed line); (b) The propagation of the light beam with an input intensity profile described by $|\psi_1(A,\alpha,\beta,x)|^2$. Here $w_0=10$, $\mu=1/\sqrt{3}$, $A=48.257$, $\alpha=-0.348$, $\beta=0.0141$, and the degree of nonlocality $w_0/(\sqrt{3}\mu)=10$.

analogy to a perturbed harmonic oscillator we can also approximately calculate the third order soliton solution or the fourth order soliton solution and so on in the generally nonlocal case for the Gaussian-function type nonlocal response.

B. The nonlocal case of the exponential-decay type nonlocal response

As indicated, to $\psi_0(A,\alpha,\beta,x)$ and $\psi_1(A,\alpha,\beta,x)$ in the nonlocal case of the Gaussian-function type nonlocal response and to $\psi_0(A,\alpha,\beta,x)$ in the nonlocal case of the exponential-decay type nonlocal response, we have $V^{(2)}(A,\alpha,\beta,0)>0$ for generally nonlocal cases. If we define

$$\tilde{V}(x) = V_0 + \frac{1}{2\mu^4}x^2 + \alpha x^4 + \beta x^6, \qquad (52)$$

we will get $V(A,\alpha,\beta,x)\approx\tilde{V}(x)$. But as shown in Figs. 13(a) and 14(a), to $\psi_1(A,\alpha,\beta,x)$ in the nonlocal case of the exponential-decay type nonlocal response, we have $V^{(2)}(A,\alpha,\beta,0)<0$. In such a case we cannot define $1/\mu^4 = V^{(2)}(A,\alpha,\beta,0)$ and cannot define the parameters μ,α,β as those in Eqs. (20). However, as shown in Figs. 13(a) and 14(a), we still can find suitable values of A,α,β for a fixed value of μ to make $V(A,\alpha,\beta,x)\approx\tilde{V}(x)$. These suitable values of A,α,β can be calculated by solving the following coupling equations:

$$V(A,\alpha,\beta,x_0) = V_0 + \frac{1}{2\mu^4}x_0^2 + \alpha x_0^4 + \beta x_0^6, \qquad (53a)$$

$$V'(A,\alpha,\beta,x_0) = \frac{1}{\mu^4}x_0 + 4\alpha x_0^3 + 6\beta x_0^5, \qquad (53b)$$

$$V''(A,\alpha,\beta,x_0) = \frac{1}{\mu^4} + 12\alpha x_0^2 + 30\beta x_0^4, \qquad (53c)$$

where $x_0\neq 0$ and $V_0=V(A,\alpha,\beta,0)$. In Appendix B we present a fixed-point method to calculate these parameters A,α,β with Eqs. (53). In Figs. 13(b), 14(b), and 15(b) we show the propagation of lights with input intensity profiles described by $|\psi_1(A,\alpha,\beta,x)|^2$. Even when the $w_0/(\sqrt{3}\mu)=0.5$, there still exists a second order nonlocal soliton. As shown by the values of $\bar{\theta}$ in Table I, $\psi_1(A,\alpha,\beta,x)$ can describe the generally nonlocal soliton state in high quality. Since the difference between $\psi_1(A,\alpha,\beta,x)$ and $\psi_1(A,0,0,x)$ is small, we can approximately get

$$\begin{aligned} V(x) &\approx -\int_{-\infty}^{+\infty}\frac{1}{2w_0}\exp\left(-\frac{|x-\xi|}{w_0}\right)|\psi_1(A,0,0,\xi)|^2 d\xi \\ &= \frac{A^2}{8w_0}\left(\frac{\mu^2}{w_0^2}+2\right)e^{\frac{\mu^2}{4w_0^2}}\left\{e^{-x/w_0}\left[\operatorname{erf}\left(\frac{\mu}{2w_0}-\frac{x}{\mu}\right)-1\right]\right. \\ &\quad \left.+e^{x/w_0}\left[\operatorname{erf}\left(\frac{\mu}{2w_0}+\frac{x}{\mu}\right)-1\right]\right\} + \frac{A^2\mu}{2\sqrt{\pi}w_0^2}e^{-x^2/\mu^2}. \end{aligned} \qquad (54)$$

By defining

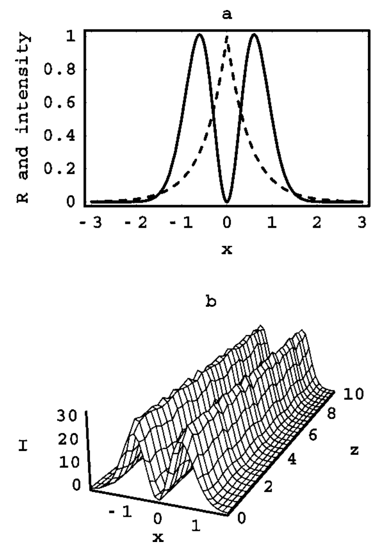

FIG. 15. (a) The comparison between $R(x)$ and $|\psi_1(A,\alpha,\beta,x)|^2$. Dashed line: $R(x)/R(0)$; solid line: $|\psi_1(A,\alpha,\beta,x)|^2/|\psi_1(A,\alpha,\beta,0.65)|^2$; (b) the propagation of the light beam with an input intensity profile described by $|\psi_1(A,\alpha,\beta,x)|^2$. Here $w_0=0.5$, $\mu=1/\sqrt{3}$, $A=6.303$, $\alpha=-0.609$, $\beta=0.0270$, and the degree of nonlocality $w_0/(\sqrt{3}\mu)=0.5$.

$$U(x) = V(x)/A^2 \qquad (55)$$

and combining with Eqs. (53), we obtain

$$A \approx \sqrt{\frac{4x_0^2/\mu^4}{24[U(x_0)-U(0)]-9U'(x_0)x_0+U''(x_0)x_0^2}}. \qquad (56)$$

For example, when $w_0=10$, $\mu=1/\sqrt{3}$, $x_0=2$, from Eqs. (54)–(56) we get $A\approx 47.323$ that is close to the numerically

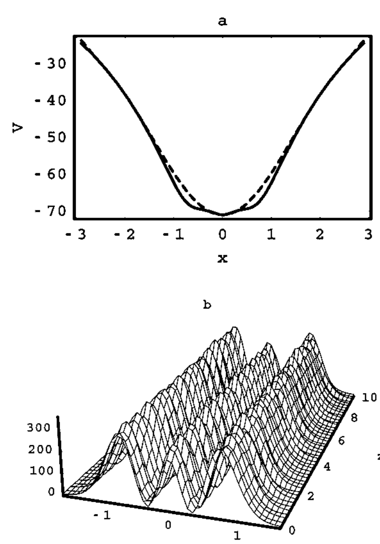

FIG. 16. (a) The comparison between $V(A,\alpha,\beta,x)$ (solid line) and $\tilde{V}(x)$ (dashed line); (b) the propagation of the light beam with an input intensity profile described by $|\psi_2(A,\alpha,\beta,x)|^2$. Here $w_0=2$, $\mu=1/\sqrt{5}$, $A=19.694$, $\alpha=-1.327$, $\beta=0.0606$, and the degree of nonlocality $w_0/(\sqrt{5}\mu)=2$.

calculated value $A=48.257$.

While $V(A,\alpha,\beta,x)\approx\tilde{V}(x)$, as shown in Figs. 13(a) and 14(a) there still exists a difference of $H(x)=V(A,\alpha,\beta,x)-\tilde{V}(x)$. To achieve higher accuracy we should take $H(x)$ into account and set $\tilde{V}(x)=V_0+x^2/(2\mu^2)+\alpha x^4+\beta x^6+H(x)$. Viewing $H(x)$ as a perturbation we will obtain another higher accurate second order soliton solution. However, the form of $H(x)$ is rather complex and we will leave it for future further work and do not intent to deal with the effect of $H(x)$ in this paper.

Now let us consider the third order nonlocal soliton. The third order generally nonlocal soliton solution in the second approximation is given by

$$\psi_2(A,\alpha,\beta,x)\approx A\left(\frac{1}{\pi\mu^2}\right)^{1/4}\exp\left(-\frac{x^2}{2\mu^2}\right)\frac{1}{\sqrt{2}}\left[-1+\frac{2x^2}{\mu^2}+\alpha\left(-\frac{45\mu^6}{16}+\frac{123\mu^4}{8}x^2-\frac{13\mu^2}{4}x^4-\frac{1}{2}x^6\right)+\alpha^2\left(\frac{11\,927\mu^{12}}{512}-\frac{24\,587\mu^{10}}{256}x^2+\frac{41\mu^8}{64}x^4+\frac{193\mu^6}{96}x^6+\frac{97\mu^4}{96}x^8+\frac{\mu^2}{16}x^{10}\right)+\beta\left(-\frac{655\mu^8}{32}+\frac{1405\mu^6}{16}x^2-\frac{125\mu^4}{8}x^4-\frac{25\mu^2}{12}x^6-\frac{1}{3}x^8\right)\right]. \tag{57}$$

As shown in Fig. 16 and Table I, $\psi_1(A,\alpha,\beta,x)$ can describe the third order generally nonlocal soliton only qualitatively. To obtain a higher more accurate third order soliton solution we should take all perturbation into account or develop another new method.

IV. CONCLUSION

In analogy to a perturbed harmonic oscillator, we calculate the fundamental and some other higher order soliton solutions in the second approximation in the generally nonlocal case. Numerical simulations confirm that the soliton solutions in the second perturbation can describe the fundamental and second order soliton states of the NNLSE (1) in high quality. For the nonlocal case of the exponential-decay type nonlocal response, the Gaussian-function-like soliton solution cannot describe the fundamental soliton state of the NNLSE (1) exactly even in the strongly nonlocal case, that greatly differs from the nonlocal case of the Gaussian-function type nonlocal response. The functional dependence of the nonlocal soliton's power and phase constant on its beamwidth are greatly different from that of the local soliton. In the strongly nonlocal case, the soliton's power and phase constant are both in inverse proportion to the fourth power of its beamwidth for the nonlocal case of the Gaussian function type nonlocal response, and are both in inverse proportion to the third power of its beamwidth for the nonlocal case of the exponential-decay type nonlocal response.

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APPENDIX A: HOW TO CALCULATE THE PARAMETERS A,α,β WITH EQ. (20)

In principle, $V(A,\alpha,\beta,x)$ and the parameters A,α,β can be found by solving Eq. (19) and Eqs. (20) directly, but these tasks are considerably involved. Here we present a fixed-point method to calculate these parameters A,α,β for a fixed value of μ. Firstly corresponding to $V(A,\alpha,\beta,x)$ in Eq. (19), we define

$$U(\alpha,\beta,x)=\frac{V(A,\alpha,\beta,x)}{A^2}. \tag{A1}$$

For an arbitrary pair of initial values of α_0,β_0 with suitable order of the magnitude, we can calculate $U(\alpha_0,\beta_0,x)$. Let

$$A_1=\sqrt{\frac{1}{\mu^4U^{(2)}(\alpha_0,\beta_0,0)}}, \tag{A2}$$

$$\alpha_1=A_1^2U^{(4)}(\alpha_0,\beta_0,0)/4!, \tag{A3}$$

$$\beta_1=A_1^2U^{(6)}(\alpha_0,\beta_0,0)/6!. \tag{A4}$$

For such a pair of values of α_1,β_1, we can find another $U(\alpha_1,\beta_1,x)$. Again we obtain another set of values $\{A_2,\alpha_2,\beta_2\}$. Repeating these steps of calculations, we can obtain a series of sets of values $\{A_2,\alpha_3,\beta_3\}$, $\{A_3,\alpha_4,\beta_4\}$, and so on. The difference between $\{A_m,\alpha_m,\beta_m\}$ and $\{A_{m+1},\alpha_{m+1},\beta_{m+1}\}$ will approach zero as the number of m approaches infinity. To some accuracy, we can calculate parameters A,α,β for a fixed value of μ.

APPENDIX B: HOW TO CALCULATE THE PARAMETERS A,α,β WITH EQ. (53)

For a fixed value of μ and one suitable point $x_0\neq 0$ (in this paper we set $x_0=2$), corresponding to $V(A,\alpha,\beta,x)$ in Eq. (19) we define

$$U(x) = \frac{V(A,\alpha,\beta,x)}{A^2}. \tag{B1}$$

For an arbitrary pair of initial values of α_0, β_0 with suitable order of the magnitude, we can calculate $U(x)$. Let

$$A_1 = \sqrt{\frac{4x_0^2/\mu^4}{24[U(x_0) - U(0)] - 9U'(x_0)x_0 + U''(x_0)x_0^2}}, \tag{B2}$$

$$\alpha_1 = A_1^2 \frac{7U'(x_0)x_0 - 12[U(x_0) - U(0)] - U''(x_0)x_0^2}{4x_0^4}, \tag{B3}$$

$$\beta_1 = A_1^2 \frac{U''(x_0)x_0^2 + 8[U(x_0) - U(0)] - 5U'(x_0)x_0}{8x_0^6}. \tag{B4}$$

For such a pair of values of α_1, β_1, we can find another $U(x)$. Again we obtain another set of values $\{A_2, \alpha_2, \beta_2\}$. Repeating the steps of the calculations, to some accuracy we can calculate parameters A, α, β for a fixed value of μ.

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